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CONGESTION MODELS AND WEIGHTED BAYESIAN POTENTIAL GAMES

ABSTRACT. Games associated with congestion situations *à la* Rosenthal (1973) have pure Nash equilibria. This result implicitly relies on the existence of a potential function. In this paper we provide a characterization of potential games in terms of coordination games and dummy games. Second, we extend Rosenthal's congestion model to an incomplete information setting, and show that the related Bayesian games are potential games and therefore have pure Bayesian equilibria.

KEY WORDS: Congestion situations, decision under uncertainty, game theory, potentials.

1. INTRODUCTION

The situation in which different agents make use of the same set of facilities and where the costs of use are expressed in terms of a function depending on the number of users has been described by Rosenthal (1973). He also showed that the associated strategic game has a pure strategy Nash equilibrium. This result is implicitly due to the existence of a potential function for this class of games, as has been shown by Monderer and Shapley (1996).

In this paper we first derive a characterization of (weighted) potential games in terms of coordination and dummy games, which enables us to compute the dimension of the linear space of weighted potential games. In the second part we propose a generalization of Rosenthal's model, which gives the possibility to model broader classes of economic and real life situations. In fact we consider situations with incomplete information, in which an agent can be of several types and has, according to each type, a specific goal. On the other hand we will allow the different individuals to have different cost functions, introducing a vector of weights. A weighted congestion model has also been proposed by Milchtaich (1996) but, as will be shown later, the role of the weight vector in our model is quite different.

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It turns out that the congestion games associated with weighted Bayesian congestion situations are Bayesian potential games and, under the common prior assumption, this implies the existence of a pure Bayesian equilibrium (van Heuman, Peleg, Tijs and Borm, 1996). These results are illustrated by a booking game. The paper concludes with an example which shows that Bayesian potential games need not to have a pure Bayesian equilibrium when the common prior assumption (Harsanyi 1967–68) is violated. This was posed as an open question by van Heumen *et al*. (1996).

2. POTENTIAL GAMES

In this section we provide a new characterization of weighted potential games, which were introduced by Monderer and Shapley (1996). As a result of this characterization by means of coordination and dummy games, the dimension of the class of potential games is easily calculated.

2.1. *A Characterization of Weighted Potential Games*

Let $G = \langle N, \{A_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$ be a game in strategic form, where N is the finite set of players, A_i is the finite set of actions available to player i and $u_i : \prod_{i\in N} A_i \to \mathbb{R}$ is some von Neumann–Morgenstern utility function for player i. The game G is called a *weighted potential game* if there exists a function $P : \prod_{i\in N} A_i \to \mathbb{R}$ and a vector $w \in \mathbb{R}^N_{++}$ such that

$$u_i(a_i, a_{-i}) - u_i(a'_i, a_{-i}) = (P(a_i, a_{-i}) - P(a'_i, a_{-i}))w_i$$

for all $i \in N, a_i \in A_i, a'_i \in A_i$ and $a_{-i} \in A_{-i} := \prod_{j\in N\setminus\{i\}} A_j$.

We now consider the following two families of games: Γ_{WC} and Γ_D. Let Γ_{WC} be the class of strategic form games $G = \langle N, \{A_i\}_{i\in N}, \{c_i\}_{i\in N}\rangle$ for which the utility function of player i is such that there exist a vector $w \in \mathbb{R}^N_{++}$ and a function $P : \prod_{i\in N} A_i \to \mathbb{R}$ with for each $i \in N : c_i = w_i P$. Such games are called the *weighted coordination games*.

Let Γ_D be the class of strategic form games $G = \langle N, \{A_i\}_{i\in N}, \{d_i\}_{i\in N}\rangle$ in which the utility function of a player does not depend on his own actions. So, for each $a_{-i} \in A_{-i}$, there exists a $k \in \mathbb{R}$

such that $d_i(a_i, a_{-i}) = k$ for each $a_i \in A_i$. These games are called *dummy games*.

In the following theorem we will use the above notions to characterize the class of weighted potential games.

THEOREM 2.1. $G = \langle N, \{A_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$ *is a weighted potential game if and only if*

$$u_i = c_i + d_i$$

for all $i \in N$ *where* c_i *and* d_i *are such that* $\langle N, \{A_i\}_{i\in N}, \{c_i\}_{i\in N}\rangle \in \Gamma_{WC}$ *and* $\langle N, \{A_i\}_{i\in N}, \{d_i\}_{i\in N}\rangle \in \Gamma_D$.[1]

Proof. We will just prove the 'only if' part. Let $G = \langle N, \{A_i\}_{i\in N}, \{c_i\}_{i\in N}\rangle$ be a weighted potential game, then there exist $w \in \mathbb{R}^N_{++}$ and $P : \prod_{i\in N} A_i \to \mathbb{R}$ with:

$$u_i(a_i, a_{-i}) = w_i P(a_i, a_{-i}) + u_i(a'_i, a_{-i}) - w_i P(a'_i, a_{-i})$$

for all $i \in N, a_i \in A_i, a'_{-i} \in A_i$ and $a_{-i} \in A_{-i}$.

Taking $c_i(a_i, a_{-i}) = w_i P(a_i, a_{-i})$ and $d_i(a_i, a_{-i}) = u_i(a_i, a_{-i}) - w_i P(a_i, a_{-i})$, it follows that $\langle N, \{A_i\}_{i\in N}, \{c_i\}_{i\in N}\rangle$ is a coordination game and $\langle N, \{A_i\}_{i\in N}, \{d_i\}_{i\in N}\rangle$ is a dummy game since $u_i(a_i, a_{-i}) - w_i P(a_i, a_{-i}) = u_i(a'_i, a_{-i}) - w_i P(a'_i, a_{-i})$ for all $i \in N, a_i \in A_i, a'_i \in A_i$ and $a_{-i} \in A_{-i}$. ■

EXAMPLE 2.1. In the following 2×2 game, which is a simplified version of Rousseau's stag-hunt game[2] (1971), a player has to decide whether to cooperate to hunt a stag (action S) or to go off on his own and hunt rabbits (action R).

$$\begin{array}{c} \\ S \\ R \end{array} \begin{array}{c} \begin{array}{cc} S & R \end{array} \\ \begin{bmatrix} 10, 20 & 0, 6 \\ 3, 0 & 3, 6 \end{bmatrix} \end{array}$$

If the weight vector is $w = (1, 2)$, then a weighted potential exists and is given by

$$P = \begin{bmatrix} 10 & 3 \\ 3 & 6 \end{bmatrix}$$

For player 1 the payoff matrix is

$$\underset{\text{w-pot. game}}{\begin{bmatrix} 10 & 0 \\ 3 & 3 \end{bmatrix}} = 1 \underset{\text{w-coord. game}}{\begin{bmatrix} 10 & 3 \\ 3 & 6 \end{bmatrix}} + \underset{\text{dummy game}}{\begin{bmatrix} 0 & -3 \\ 0 & -3 \end{bmatrix}}$$

and likewise for player 2

$$\underset{\text{w-pot. game}}{\begin{bmatrix} 20 & 6 \\ 0 & 6 \end{bmatrix}} = 2 \underset{\text{w-coord. game}}{\begin{bmatrix} 10 & 3 \\ 3 & 6 \end{bmatrix}} + \underset{\text{dummy game}}{\begin{bmatrix} 0 & 0 \\ -6 & -6 \end{bmatrix}}$$

2.2. *On the Dimension of the Linear Space of Potential Games*

Consider the family $\Gamma^{N,m}$ of strategic form games with fixed player set $N = \{1, \ldots, n\}$ and fixed action space $A = \prod_{i \in N} A_i$ with $m_i = |A_i|$ and $m = (m_1, \ldots, m_n)$. Clearly the family $\Gamma^{N,m}$ can be identified with the function space $(\mathbb{R}^N)^{\prod_{i \in N} A_i}$ of maps from $\prod_{i \in N} A_i$ into $\mathbb{R}^N$ in a natural sense, according to the fact that the game is 'known' if for every action profile $a \in \prod_{i \in N} A_i$ the utility vector $(u_1(a), u_2(a), \ldots, u_n(a))$ is given. Therefore we have that for the family $\Gamma^{N,m}$

$$\dim(\Gamma^{N,m}) = \dim(\mathbb{R}^N)^{\prod_{i \in N} A_i} = n \prod_{i \in N} m_i.$$

In Theorem 2.1 we have characterized (weighted) potential games as the sum of coordination games and dummy games. Using that result, we will derive the dimension of the linear space of potential games.

Let $P\Gamma^{N,m} \subset \Gamma^{N,m}$ denote the subclass of potential games with N players and $m = (m_1, \ldots, m_n)$, where $m_i = |A_i|$. As a corollary of Theorem 2.1 we have that

$$(*) \qquad P\Gamma^{N,m} = \Gamma_D^{N,m} + \Gamma_C^{N,m}$$

where $\Gamma_C^{N,m}$ is the class of *coordination games* and $\Gamma_D^{N,m}$ is the class of *dummy games*.

We can now prove the following

THEOREM 2.2. *For the linear space of potential games $P\Gamma^{N,m}$:*

$$\textit{dim } P\Gamma^{N,m} = \prod_{i=1}^{n} m_i + \sum_{i=1}^{n} \left(\prod_{j \neq i} m_j \right) - 1.$$

Proof. Because of $(*)$ we have that $\dim(P\Gamma^{N,m}) = \dim(\Gamma_D^{N,m}) - \dim(\Gamma_C^{N,m} \cap \Gamma_D^{N,m})$. The dimensions of the right hand side of the equation can be easily computed, identifying $\Gamma_C^{N,m}$ with the function space $(\mathbb{R})^{\prod_{i \in N} A_i}$, and $\Gamma_D^{N,m}$ with the function space $(\mathbb{R})^{\prod_{i \neq 1} A_i} \times \cdots \times (\mathbb{R})^{\prod_{i \neq n} A_i}$.

Then $\dim((\mathbb{R})^{\prod_{i \in N} A_i}) = \prod_{i \in N} m_i$ and $\dim((\mathbb{R})^{\prod_{i \neq 1} A_i} \times \cdots \times (\mathbb{R})^{\prod_{i \neq n} A_i}) = \sum_{i \in N} \prod_{j \neq i} m_j$.

Now it suffices to show that $\dim(\Gamma_D^{N,m} \cap \Gamma_C^{N,m}) = 1$. Using the definition of coordination and dummy games a game $\langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ in $(\Gamma_D^{N,m} \cap \Gamma_C^{N,m})$ has the property that there exist $u : \prod_{i \in N} A_i \to \mathbb{R}$ such that $u_i(a) = u(a)$ for all $i \in N, a \in \prod_{i \in N} A_i$ because it is a coordination game and $u(a) = u(b)$ for all $a, b \in \prod_{i \in N} A_i$ since it also is a dummy game. It means that $(\Gamma_D^{N,m} \cap \Gamma_C^{N,m})$ can be identified with $\mathbb{R}$. ■

REMARK 2.3. It should be mentioned that Monderer and Shapley (1996) provide such note on dimensions without proof in Appendix B of their paper. It is straightforward to show that the same result holds even in the computation of the dimension of the linear space of weighted potential games with fixed weight vector.

3. CONGESTION SITUATIONS AND BAYESIAN POTENTIAL GAMES

Rosenthal (1973) considers congestion situations where each agent wants to achieve an individual objective by choosing a suitable subset of a set M of common facilities. The using cost of each separate facility depends on the number of users.

Congestion situations give rise to potential games and, conversely, each finite potential game can be derived from a congestion situation (Monderer and Shapley, 1996). An important property of potential games is the existence of a pure Nash equilibrium. In this section we look at a general type of congestion situation which gives rise to Bayesian potential games with pure Bayesian equilibria. Our congestion model constitutes a generalization of Rosenthal's one.

EXAMPLE 3.1. (A booking game). Consider the situation in which two agents, depending on their types, want to go to a concert hall to see Verdi's Aida (type κ) or to the stadium (type σ), to watch the soccer match Ajax–AC Milan. There are three places in which the tickets can be booked, the stadium (S), where it is possible to buy

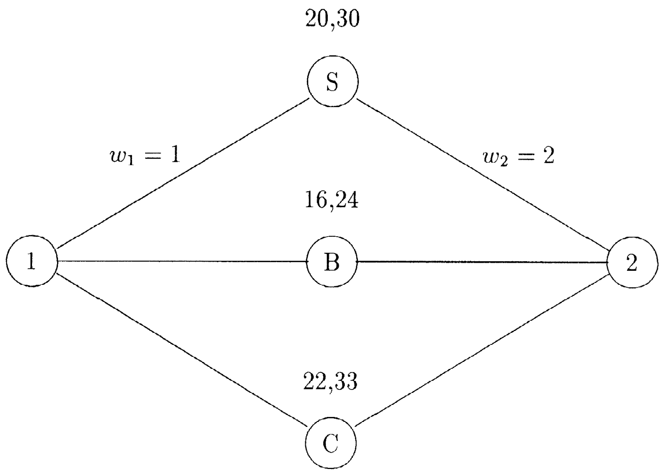

Fig. 1.

only the soccer ticket, the concert hall (C), which only sells opera's tickets and a booking office (B) where both tickets can be obtained. In order to reserve a ticket, the agents decide to call the providing facilities. We assume that the first agent is at home while the second one is on the street and therefore has to call from a public telephone. If both call the same facility simultaneously, then someone has to wait on line. For this reason, we assume that the average calling time is increased by 50%, which obviously leads to an additional cost. Finally, if the agents call the 'correct' facility with respect to their type, then their utility is increased by some reward. The picture in Figure 1 illustrates this situation. In particular, we assume that the average time for calling the stadium is 20 if only one agent calls and 30 if both try to ring. For the booking office we similarly have an average calling time of 16 and 24 and for the concert hall of 22 and 33, respectively. Moreover, we take into account the fact that agent 1 calls from home while agent 2 calls from a public telephone. Since in the Netherlands the call cost per unit of time from a public phone is twice as high as the cost for calling from home, we have to 'weight' the average calling time by a factor of one for agent 1 and two for agent 2.

Furthermore we suppose that agent 1 can enjoy entertainments more than agent 2 and therefore their rewards for obtaining the right ticket, independently of their type, are 500 and 400, respectively. Obtaining the wrong ticket gives a zero reward.

The general model underlying this kind of situation is called a *weighted Bayesian congestion situation* and can be described as follows:

$$[N, M, \{T_i\}_{i\in N}, p, \{r_i\}_{i\in N}, \{c_k\}_{k\in M}, w]$$

where

- $N = \{1, 2, \ldots, n\}$ is the finite set of *players*.
- $M = \{1, 2, \ldots, m\}$ is the finite set of *facilities*.
- T_i is the finite set of *types* of users $i \in N$, which specify the goal of each player.
- $p \in \Delta(T)$ is a *probability measure* on $T := \prod_{i\in N} T_i$.
- $r_i : 2^M \times T_i \to \mathbb{R}$; $r_i(a_i, t_i)$ is the *reward* of player i for using the facilities in $a_i \in 2^M$ if his type is t_i.
- $c_k : \{0, 1, \ldots, |N|\} \to \mathbb{R}_+$ is the *cost function* depending on the number of users of facility k.
- $w \in \mathbb{R}^N_{++}$ is a interpreted as follows: player i has costs $w_i c_k(\ell)$, $\ell \in \{0, 1, \ldots, |N|\}$ for factor k if there are ℓ users.

We are now going to define a Bayesian game (with common prior) corresponding to the weighted congestion situation described above. The general form of a Bayesian game G is given by

$$G = \langle N, \{A_i\}_{i\in N}, \{T_i\}_{i\in N}, p, \{u_i\}_{i\in N}\rangle$$

where N, $\{T_i\}_{i\in N}$ and p play the obvious roles and the set of actions is defined by $A_i := 2^M$ for all players $i \in N$ and the utility function $u_i : (2^M)^N \times T \to \mathbb{R}$ for all $i \in N$ by

$$(**) \qquad u_i(a, t) = r_i(a_i, t_i) - w_i \sum_{k\in a_i} c_k(n_k(a_1, \ldots, a_n))$$

for all $s \in (2^M)^N$ and $t \in T$, where $n_k(a_1, \ldots, a_n)$ is the number of users of facility k according to the chosen facility sets. It means that in our model the role of the weights is to extend Rosenthal's framework allowing different cost functions for each player. The problem of how to model a players specific contribution to the congestion has been considered also by Milchtaich in a recent paper (1996). In his framework however, where the weights are used to model the fact that a car and a heavy truck play different roles in inducing congestion, it is impossible to guarantee the existence of a pure strategy Nash equilibrium if not all weights are equal.

Formally, given a Bayesian game $G = \langle N, \{A_i\}_{i \in N}, p, \{u_i\}_{i \in N} \rangle$ a strategy of player i is a map $x_i : T_i \to A_i$. A strategy profile $x \in X := \prod_{i \in N} X_i$ is called a (pure) *Bayesian equilibrium* of the game G if for all $i \in N, t_i \in T_i$ and $a_i \in A_i$:

$$\sum_{t_{-i} \in T_{-i}} p(t_{-i}|t_i) u_i(\{x_j(t_j)\}_{j \in N}, t) \geqslant \sum_{t_{-i} \in T_{-i}} p(t_{-i}|t_i) u_i((\{x_j(t_j)\}_{j \in N \setminus \{i\}}, a_i), t)$$

where $p(t_{-i}|t_i)$ is the conditional probability[3] player i puts on t_{-i}, assuming that his own type is t_i.

For a Bayesian game $G = \langle N, \{A_i\}_{i \in N}, \{T_i\}_{i \in N}, p, \{u_i\}_{i \in N} \rangle$ the corresponding *ex ante* game $\hat{G}$ is defined by

$$\hat{G} = \langle X_1, \ldots, X_n, \hat{u}_1, \ldots, \hat{u}_n \rangle$$

where for all $i \in N, X_i = (A_i)^{T_i}$ is the strategy set for player i and $\hat{u}_i(x) = \sum_{t \in T} p(t) u_i((x_j(t_j))_{i \in N}, t)$ is the payoff function.

Harsanyi (1968, II, p. 321) proved the following theorem.

THEOREM 3.1. *For any Bayesian game G with common prior, x is a Bayesian equilibrium of G if and only if x is a Nash equilibrium of the ex ante game $\hat{G}$.*

In Theorem 3.4 it will be shown that the game associated to a weighted Bayesian congestion situation is a weighted Bayesian potential game in the sense of the following:

DEFINITION 3.1. Let G be a Bayesian game. G is called a weighted Bayesian potential game if there exist a function $q : A \times T \to \mathbb{R}$ and a vector $w \in \mathbb{R}^N_{++}$ such that, for every $i \in N, a \in A, b_i \in A_i$, and $t \in T$

$$u_i(a, t) - u_i((a_{-i}, b_i), t) = w_i(q(a, t) - q((a_{-i}, b_i), t)).$$

The function q is called a weighted potential for G.

THEOREM 3.2. *Let* $G = \langle N, \{A_i\}_{i\in N}, \{T_i\}_{i\in N}, p, \{u_i\}_{i\in N}\rangle$ *be a weighted Bayesian game arising from a weighted Bayesian congestion situation* $[N, M, \{T_i\}_{i\in N}, p, \{r_i\}_{i\in N}, \{c_k\}_{k\in M}, w]$. *Then* G *is a weighted Bayesian potential game.*

Proof. Define

$$q(a,t) = \sum_{i\in N} \frac{r_i(a_i,t_i)}{w_i} - \sum_{k\in M} \sum_{\ell=0}^{n_k(a)} c_k(\ell).$$

Then, using $(**)$

$$w_i q(a,t) - u_i(a,t) \;=\; -\sum_{j\neq i} \frac{r_j(a_j,t_j)w_i}{w_j} \\ -w_i \sum_{k\in M} \sum_{\ell=0}^{n_k(a_{-i})} c_k(\ell).$$

This means that $w_i q(a,t) - u_i(a,t)$ does not depend on the action a_i. Therefore q is a weighted potential for G. ■

It is not always the case that a Bayesian potential game can be derived from a weighted Bayesian congestion situation. Igal Milchtaich provided the following counterexample. Consider a Bayesian potential game where $N = \{1,2\}, T_1 = \{\alpha\}, T_2 = \{\gamma, \delta\}$. The payoff matrices are

$$\begin{array}{ccc} & \gamma & \delta \\ & \begin{array}{cc} L & R \end{array} & \begin{array}{cc} L & R \end{array} \\ \alpha \quad \begin{array}{c} T \\ B \end{array} & \begin{bmatrix} 1,1 & 0,0 \\ 0,0 & 1,1 \end{bmatrix} & \begin{bmatrix} 0,0 & 1,1 \\ 1,1 & 0,0 \end{bmatrix} \end{array}$$

Recalling equation $(**)$, the following should be true

$$u_2((a_1,L),(\alpha,\gamma)) - u_2((a_1,L),(\alpha,\delta)) \\ = r_2(L,\gamma) - r_2(L,\delta).$$

In other words, the difference should not depend on the action taken by player 1. Going back to the example, it is easy to see that this difference is plus or minus 1 depending on player’s choice.

REMARK 3.3. Our result can be easily extended to type dependent weights, but this could give rise to interpretation problems (see the booking game for an example). Therefore we prefer to confine all the private information present in our model to the type dependent targets.

We now apply the previous results to our example of a booking situation and obtain the associated *booking game*, which is described as follows:

$$\langle \{1,2\}, \{S,B,C\}, \{\sigma,\kappa\}, p, \{u_1,u_2\}\rangle$$

where the common prior p will be specified later and u_1, u_2 are the utility functions of the players. For the sake of simplicity, we eliminate the strategies which suggest to each player to use two or three different facilities, because they are obviously dominated. Therefore the knotted payoff matrices are, depending on the type of each player,

$$\begin{array}{cc} & \begin{array}{ccc|ccc} & \kappa & & & \sigma & \\ S & B & C & S & B & C \end{array} \\ \kappa \begin{array}{c} S \\ B \\ C \end{array} & \left[\begin{array}{ccc} (-30,-60) & (-20,368) & (-20,356) \\ (484,-40) & (476,352) & (484,356) \\ (478,-40) & (478,368) & (467,334) \end{array}\right] \left[\begin{array}{ccc} (-30,340) & (-20,368) & (-20,-44) \\ (484,360) & (476,352) & (484,-44) \\ (478,360) & (478,368) & (467,-66) \end{array}\right] \\ \sigma \begin{array}{c} S \\ B \\ C \end{array} & \left[\begin{array}{ccc} (470,-60) & (480,368) & (480,356) \\ (484,-40) & (476,352) & (484,356) \\ (-22,-40) & (-22,368) & (-33,334) \end{array}\right] \left[\begin{array}{ccc} (470,340) & (480,368) & (480,-44) \\ (484,360) & (476,352) & (484,-44) \\ (-22,360) & (-22,368) & (-33,-66) \end{array}\right]. \end{array}$$

In other words, for example, player 1 likes to go to the concert hall (i.e. he is of type κ) and player 2 is fond of soccer (type σ), then, when both call the booking office to book their ticket, player 1 has a utility of $476 = 500 - 24$ and player 2 of $352 = 400 - 2 * 24$. We can compute now an associated potential, which is given by

$$\begin{array}{cc} & \begin{array}{ccc|ccc} & \kappa & & & \sigma & \\ S & B & C & S & B & C \end{array} \\ \kappa \begin{array}{c} S \\ B \\ C \end{array} & \left[\begin{array}{ccc} 0 & 214 & 208 \\ 514 & 710 & 712 \\ 508 & 712 & 695 \end{array}\right] \left[\begin{array}{ccc} 192 & 206 & 0 \\ 706 & 702 & 504 \\ 700 & 704 & 487 \end{array}\right] \end{array}$$

$$\sigma \quad \begin{matrix} S \\ B \\ C \end{matrix} \begin{bmatrix} 492 & 706 & 700 \\ 506 & 702 & 704 \\ 0 & 204 & 187 \end{bmatrix} \begin{bmatrix} 705 & 719 & 513 \\ 719 & 715 & 517 \\ 213 & 217 & 0 \end{bmatrix}.$$

Consider now the following (common) prior

$$p = \begin{matrix} \\ \kappa \\ \sigma \end{matrix} \begin{matrix} \kappa & \sigma \\ \left[\frac{4}{10}\right. & \left.\frac{2}{10}\right] \\ \left[\frac{1}{10}\right. & \left.\frac{3}{10}\right] \end{matrix}.$$

It turns out that several strategies are dominated for both players. So the ‘knotted’ *ex ante* game is:

	BS	BB	CS	CB
BS	$476.2, 351.6$	$477.6, 358.4$	$479.4, 352$	$480.8, 358.8^*$
BB	$480, 356$	$476, 352$	$484, 358^*$	$480, 354$
CS	$475.8, 358$	$478.8, 368^*$	$471.4, 343.2$	$474.4, 353.2$
CB	$479.6, 362.4^*$	$477.2, 361.6$	$476, 349.2$	$473.6, 348.4$

There are four Nash equilibria in pure strategies and considering an associated *ex ante* potential matrix, we can show that (BB, CS) is the potential maximizer.

	BS	BB	CS	CB
BS	707.3	710.7	707.5	710.9^*
BB	711.1	709.1	712.1^+	710.1
CS	706.9	711.9^*	699.5	704.5
CB	710.7^*	710.3	704.1	703.7

In terms of our congestion situation, the behaviour prescribed by the potential maximizer is that if player 2 is of type κ, he has to call the concert hall directly, while if he is of type σ, he has to call the stadium directly. Player 1 instead, regardless of his type, must always call the booking office in order to get the desired tickets.

4. INCONSISTENT PRIORS

In this paper we have considered a weighted congestion model, which has been associated to a weighted Bayesian potential game. It is well known (see van Heuman *et al*. 1996) that every Bayesian potential game with common prior has a pure strategy equilibrium. In the same paper the problem whether each Bayesian potential game has a pure equilibrium is posed as an open question. It turns out that this need not be the case, as can be seen in the next example. To show our result we first state the following:

LEMMA 4.1. *Let* $G = \langle N, \{A_i\}_{i\in N}, \{T_i\}_{i\in N}, \{p_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$ *be a general Bayesian game where* p_i *is the probability measure of player* i *over* $T := \prod_{i\in N} T_i$. *Let* $\hat{G}$ *be the game associated with* G *where, for every* $i \in N$, X_i *is the set of pure strategies of player* i *and for every* $x \in X, i \in N$,

$$\hat{u}_i(x) := \sum_{t\in T} p_i(t)u_i(\{x_j(t_j)\}_{j\in N}, t)$$

then if $\{x_i\}_{i\in N}$ *is a Bayesian equilibrium of* G, $\{x_i\}_{i\in N}$ *is a Nash equilibrium of the ex-ante game* $\hat{G}$ *associated with* G.

Proof. By definition of a Bayesian equilibrium, we have that for all $t_i \in T_i$, $a_i \in A_i$,

$$\sum_{t_{-i}} p_i(t_{-i}|t_i)u_i(\{x_j(t_j)\}_{j\in N}, t) \geqslant \sum_{t_{-i}} p_i(t_{-i}|t_i)u_i((\{x_j(t_j)\}_{j\in N\setminus\{i\}}, a_i), t).$$

Then

$$\sum_{t_{-i}} \left(\sum_{s_{-i}} p_i(s_{-i}, t_i)\right) p_i(t_{-i}|t_i)u_i(\{x_j(t_j)\}_{j\in N}, t) \geqslant \sum_{t_{-i}} \left(\sum_{s_{-i}} p_i(s_{-i}, t_i)\right) p_i(t_{-i}|t_i)u_i((\{x_j(t_j)\}_{j\in N\setminus\{i\}}, a_i), t)$$

so for each $t_i \in T_i$, $a_i \in A_i$

$$\sum_{t_{-i}} p_i(t)u_i(\{x_j\}_{j\in N}, t) \geqslant \sum_{t_{-i}} p_i(t)u_i((\{x_j\}_{j\in N\setminus\{i\}}, a_i), t).$$

This means that

$$\sum_{t} p_i(t)u_i(\{x_j(t_j)\}_{j\in N}, t) \geqslant \sum_{t} p_i(t)u_i((\{x_j(t_j)\}_{j\in N\setminus\{i\}}, a_i), t)$$

and thus for all $y_i \in X_i$

$$\hat{u}_i(x) \geqslant \hat{u}_i(y_i, x_{-i}). \qquad \blacksquare$$

Now we look at a specific Bayesian potential game with inconsistent priors. There are two players, 1 and 2. Each player has two different types $T_1 = \{\alpha, \beta\}, T_2 = \{\gamma, \delta\}$. The priors p_1, p_2 are given by

$$p_1 = \begin{matrix} & \gamma & \delta \\ \alpha & 0 & \frac{3}{4} \\ \beta & \frac{1}{4} & 0 \end{matrix} \qquad p_2 = \begin{matrix} & \gamma & \delta \\ \alpha & \frac{1}{3} & 0 \\ \beta & 0 & \frac{2}{3} \end{matrix}$$

and the payoff matrices are given typewise:

$$\begin{array}{ccc} & \gamma & \delta \\ & \begin{matrix} L & R \end{matrix} & \begin{matrix} L & R \end{matrix} \\ \alpha \quad \begin{matrix} T \\ B \end{matrix} & \begin{bmatrix} 1,1 & 0,0 \\ 0,0 & 1,1 \end{bmatrix} & \begin{bmatrix} 0,0 & 1,1 \\ 1,1 & 0,0 \end{bmatrix} \\ \beta \quad \begin{matrix} T \\ B \end{matrix} & \begin{bmatrix} 0,0 & 1,1 \\ 1,1 & 0,0 \end{bmatrix} & \begin{bmatrix} 0,0 & 1,1 \\ 1,1 & 0,0 \end{bmatrix} \end{array}$$

The corresponding $\hat{G}$ game is given by

$$\begin{matrix} & LL & LR & RL & RR \\ TT & 0, \frac{1}{3} & \frac{3}{4}, 1 & \frac{1}{4}, 0 & 1, \frac{2}{3} \\ TB & \frac{1}{4}, 1 & 1, \frac{1}{3} & 0, \frac{2}{3}, & \frac{3}{4}, 0 \\ BT & \frac{3}{4}, 0 & 0, \frac{2}{3} & 1, \frac{1}{3} & \frac{1}{4}, 1 \\ BB & 1, \frac{2}{3} & \frac{1}{4}, 0 & \frac{3}{4}, 1 & 0, \frac{1}{3} \end{matrix}$$

It is easy to show that there are no pure Nash equilibria in this game. Then using Lemma 4.1 the Bayesian game does not have a pure Bayesian equilibrium.

NOTES

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[1] Using the sum characterization it is easy to get a new axiomatic characterization of the potential maximizer, following the line originally proposed by Peleg, Potters and Tijs (1996).

[2] It is interesting to note that in 2×2 weighted stag-hunt games the potential maximizer selects the same equilibrium as the Harsanyi–Selten (1988), Güth (1992) and Carlsson–Van Damme (1993) criteria.

[3] This conditional probability can be defined if the assumption is made that every player puts positive probability on each of his types. We restrict ourselves to games for which this is the case.

REFERENCES

Carlsson, H. and van Damme, E.: 1993, 'Global games and equilibrium selection', *Econometrica* **61**, 989–1018.

Güth, W.: 1992, 'Equilibrium selection by unilateral deviation stability', in: R. Selten (ed.) *Rational Interaction. Essays in Honor of John C. Harsanyi*, Berlin: Springer Verlag.

Harsanyi, J.C.: 1967–68, 'Games with incomplete information played by "Bayesian" players I–III', *Management Science* **14**, 159–182, 320–334, 486–502.

Harsanyi, J.C. and Selten, R.: 1988, *A General Theory of Equilibrium Selection in Games*, Cambridge, MA: MIT Press.

van Heumen, R., Peleg, B., Tijs, S., and Borm, P.: 1996, 'Axiomatic Characterizations of solutions for Bayesian games', *Theory and Decision* 40, 103–130.

Milchtaich, I.: 1996, 'Congestion games with player specific payoff functions', *Games and Economic Behavior* 13, 111–124.

Monderer, D. and Shapley, L.S.: 1996, 'Potential games', *Games and Economic Behavior* 14, 124–143.

Peleg, B., Potters, J. and Tijs, S.: 1996, 'Minimality of consistent solutions for strategic games, in particular for potential games', *Economic Theory* **7**, 81–92.

Rosenthal, R.W.: 1973, 'A class of games possessing pure-strategy Nash equilibria', *International Journal of Game Theory* **2**, 65–67.

Rousseau, J.J.: 1971, *Discours sur l'origine et les fondements de l'inègalité parmi les hommes*, from *Oeuvres complètes* II. Paris: Editions du Seuil.

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